Limits and degeneracies of discrete Painlevé equations: a sequel

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Abstract

We present the discrete systems which result from the discrete Painlevé equations q-PVI and d-PV associated to the affine Weyl group E(7)1. Two different procedures ("limits" and "degeneracies") are used, giving rise to a host of new discrete Painlevé equations but also to some equations which are integrable through linearisation.

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1. Introduction

In a previous publication [1], the Capel Festschrift (referred to in what follows as CF), we analysed the discrete Painlevé equations (d-P) known at that time and

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applied two different procedures which made possible the derivation of new integrable systems. The first procedure was simply a limit of the discrete Painlevé equation when some of its parameters were taking special values (typically zero or infinity). This resulted in a d-$P$ with fewer parameters, with obvious influence on the continuous limit. The second procedure was dubbed “degeneracy” (a term which is undoubtedly unfortunate). The method is the following: Given a d-$P$ we consider its autonomous limit, namely the corresponding QRT mapping [2]. On the latter we make some assumption on the coefficients, which results to a simplification of the mapping (through cancellation of factors in the numerator and the denominator). Next, we deautonomise the resulting mapping and seek the integrable nonautonomous extensions by the application of some discrete integrability criterion (singularity confinement [3] or equivalently in the present setting, algebraic entropy [4]).

In CF, these two procedures were applied to four different families of d-$P$s: d-$PI$ = d-$PII$, q-$PIII$, d-$PIV$ and q-$PV$. Although this classification is somewhat arbitrary it still makes sense in the light of the results of Ref. [5]. (At this point, we must stress the fact that the term “classification” does not refer to that of discrete Painlevé equations obtained by Sakai [6]. It is just a convenient way to regroup the d-$P$s based on the structure of the underlying QRT system). The results of CF were complemented in several subsequent publications [7] and we presented the study of the full freedom of these d-$P$s, linking them to some affine Weyl group for the description of the geometry of their transformations, and we also worked on their bilinearisation. These studies also furnished the occasion to fill in the omissions in CF. However one important sector was still unexplored. When CF was published, the knowledge of the d-$P$s did not extend beyond q-$PV$. In particular, the discrete analogue of $PVI$ was missing. The discrete $PVI$ was discovered only later, first in an asymmetric (in the QRT terminology) form of q-$PIII$ [8] and later as q-$PVI$ [9], which also lead to the discovery of d-$PV$. The latter two equations have the form

\[
\frac{(x_n x_{n+1} - q_n q_{n+1})(x_n x_{n-1} - q_n q_{n-1})}{(x_n x_{n+1} - 1)(x_n x_{n-1} - 1)} = \frac{(x_n - q_n)(x_n - q_n/a)(x_n - b q_n)(x_n - q_n/b)}{(x_n - c)(x_n - 1/c)(x_n - d)(x_n - 1/d)}, \tag{1.1}
\]

\[
\frac{(x_n + x_{n+1} - z_n - z_{n+1})(x_n + x_{n-1} - z_n - z_{n-1})}{(x_n + x_{n+1})(x_n + x_{n-1})} = \frac{(x_n - z_n - a)(x_n - z_n + a)(x_n - z_n - b)(x_n - z_n + b)}{(x_n - c)(x_n + c)(x_n - d)(x_n + d)}, \tag{1.2}
\]

where $q_n = q_0 \lambda^n$, $z_n = z_0 + \alpha n$ and $a, b, c, d$ are free constants. The corresponding $PVI$ and $PV$ limits of (1.1) and (1.2) were presented in Ref. [9]. In the same article we obtained the “asymmetric” form of these equations. For q-$PVI$ we have the “asymmetric” form.
where \( q_{n+1/2} = q_n \sqrt{\lambda} \), \( q_{n-1/2} = q_n / \sqrt{\lambda} \), and with the constraints \( abcd = 1 \) and \( prst = 1 \). Similarly, we find the asymmetric d-PV

\[
(x_n + y_{n+1} - z_n - z_{n+1/2})(y_n + x_n - z_n - z_{n-1/2})
\]

\[
= \frac{(x_n - z_n - a)(x_n - z_n - b)(x_n - z_n - c)(x_n - z_n - d)}{(x_n - p)(x_n - r)(x_n - s)(x_n - t)},
\]  

(1.4a)

\[
(y_n + x_n - z_n - z_{n-1/2})(y_n + x_n - z_n - z_{n-1/2})
\]

\[
= \frac{(y_n - z_n - 1/2 + a)(y_n - z_n - 1/2 + b)(y_n - z_n - 1/2 + c)(y_n - z_n - 1/2 + d)}{(y_n + p)(y_n + r)(y_n + s)(y_n + t)},
\]  

(1.4b)

where \( z_{n+1/2} = z_n + \alpha / 2 \), \( z_{n-1/2} = z_n - \alpha / 2 \), and with the constraints \( a + b + c + d = 0 \) and \( p + r + s + t = 0 \).

These equations were shown to possess the singularity confinement property and to have zero algebraic entropy. Moreover, they were shown in Ref. [10] to fit into the Sakai classification of d-PVs (associated to the affine Weyl group \( E_7^{(1)} \)), a fact that further confirmed their integrability.

In this paper, we are going to study the limits and degeneracies of Eqs. (1.1) and (1.2). In the case of the limits, we shall provide the continuous limits, linking them to already known d-PVs. For the degeneracies, we shall investigate their full freedom using discrete integrability criteria [11]. An interesting result is that, just as in the case of d-PVs in the CF, not all the resulting equations are d-PVs. Three of the nonautonomous mappings have coefficients which can be expressed in terms of a free function of the independent variable. In contrast to d-PVs, these equations are not integrable by some spectral method but can be reduced to a linear difference problem [12].
2. Limits and degeneracies of the \( q \)-Painlevé VI equation

In this section, we shall derive the various integrable mappings related to the \( q \)-PV equation

\[
\frac{(x_n x_{n+1} - q_n q_{n+1})(x_n x_{n-1} - q_n q_{n-1})}{(x_n x_{n+1} - 1)(x_n x_{n-1} - 1)} = \frac{(x_n - a q_n)(x_n - q_n/a)(x_n - b q_n)(x_n - q_n/b)}{(x_n - c)(x_n - 1/c)(x_n - d)(x_n - 1/d)},
\]

(2.1)

where \( q_n = q_0^{2n} \) and \( a, b, c, d \) are the parameters of the equation. First, we remark that this equation is invariant under the following transformations:

\[
x \to 1/x, \quad q \to 1/q
\]

and also

\[
x \to q/x, \quad q \to q,
\]

provided one permutes the numerator and the denominator of the mapping in the latter case (upon exchanging \( a \) with \( c \) and \( b \) with \( d \)). We start by obtaining the limits of (2.1) when the parameters take some special value. In the process the meaning of \( q \) or of the parameters that survive does not change. By letting \( a \to \infty \) and \( c \to \infty \) simultaneously we find the equation

\[
\frac{(x_n x_{n+1} - q_n q_{n+1})(x_n x_{n-1} - q_n q_{n-1})}{(x_n x_{n+1} - 1)(x_n x_{n-1} - 1)} = f q_n \frac{(x_n - b q_n)(x_n - q_n/b)}{(x_n - d)(x_n - 1/d)},
\]

(2.2)

where \( f \) stands for the ratio \( a/c \). As it stands the equation has \( P_V \) as a continuous limit. Indeed taking \( f = 1 + \varepsilon^2 f_2, \quad b = 1 + \varepsilon^2 b_2, \quad d = 1 + \varepsilon^2 d_2, \quad q = 1 + \varepsilon \sqrt{\gamma}, \quad x = 1 + \varepsilon \sqrt{\zeta}(1 - w) \), we obtain at \( \varepsilon \to 0 \) the canonical form of \( P_V \) [13] with \( x = d_2^2/2, \beta = -b_2^2/2, \gamma = f_2/2, \delta = -1/8 \). The full freedom of this equation is obtained through

\[
\frac{(x_n x_{n+1} - q_n q_{n+1})(x_n x_{n-1} - q_n q_{n-1})}{(x_n x_{n+1} - 1)(x_n x_{n-1} - 1)} = f q_n \frac{(x_n - \theta_n b q_n)(x_n - q_n/b)}{\theta_n(x_n - d)(x_n - 1/d)}
\]

(2.3)

with \( \log \theta_n = \gamma(-1)^n \). The geometry of the transformations of this equation is related to the affine Weyl group \((\text{aWg}) D_5\). Next we implement one more limit by taking \( f \to 0 \) and \( b \to 0 \) in (2.2) simultaneously and obtain the equation

\[
\frac{(x_n x_{n+1} - q_n q_{n+1})(x_n x_{n-1} - q_n q_{n-1})}{(x_n x_{n+1} - 1)(x_n x_{n-1} - 1)} = \frac{a q_n^2 x_n}{(x_n - d)(x_n - 1/d)},
\]

(2.4)

where \( g \) now stands for the ratio \(-f/b\). This equation has \( P_{IV} \) as continuous limit, obtained by putting \( x = -1 + \varepsilon u, \quad g = 1 + \varepsilon^2 z, \quad d = -1 + \varepsilon^2 \sqrt{-\beta/2} \) and \( q = i(1 + \varepsilon t) \). Taking \( \varepsilon \to 0 \) we obtain \( P_{IV} \) for \( u \) as a function of \( t \) in canonical form [13]. As a matter of fact (2.4) is exactly the \( P_{IV} \) equation derived by Kajiwara et al. [14], defined by the mapping
Indeed from (2.5) we have that transformations can be described by the aWg $A_2^2$ does not possess any extra freedom and as shown in Ref. [14] the geometry of its section can be obtained using the transformations we presented in the beginning of this section. If we restrict ourselves to mappings of symmetric form, we obtain two equations that satisfy the integrability requirement:

Next we proceed to the study of what we called “degeneracies” of Eq. (2.1). We begin by assuming that the numerator and denominator of (2.1) have a common factor which drops out, i.e., that the equation reduces to

Eliminating $y$ and $z$ we precisely obtain (2.4) with $g = b/c$ and $d = -a$. Eq. (2.4) does not possess any extra freedom and as shown in Ref. [14] the geometry of its transformations can be described by the aWg $A_2 \times A_1$. A dual form of this mapping can be obtained using the transformations we presented in the beginning of this section

Thus we introduce $\lambda$ through $abc = \lambda$ and the independent variable through $q_n = x_n y_n z_n$, so that $q_n+1 = \lambda q_n$. Eliminating $y$ and $z$ we precisely obtain (2.4) with $g = b/c$ and $d = -a$. Eq. (2.4) does not possess any extra freedom and as shown in Ref. [14] the geometry of its transformations can be described by the aWg $A_2 \times A_1$. A dual form of this mapping can be obtained using the transformations we presented in the beginning of this section

If we restrict ourselves to mappings of symmetric form, we obtain two equations that satisfy the integrability requirement:

If we restrict ourselves to mappings of symmetric form, we obtain two equations that satisfy the integrability requirement:
The latter possesses a dual form

\[
\frac{(x_n x_{n+1} - q_n q_{n+1})(x_n x_{n-1} - q_n q_{n-1})}{(x_n x_{n+1} - 1)(x_n x_{n-1} - 1)} = \left( \frac{x_n - q_n}{x_n - 1/q_n} \right) \frac{(x_n - b q_n (x - q_n/b)}{(x_n - c)(x_n - 1/c)} .
\]  

(2.10)

We now turn to the question of the full freedom of the above equations, i.e., we allow \( q \) and the parameters to have binary, ternary or higher degrees of freedom. In relation to (2.8) we find that (2.7) with \( j^2 = 1 \) log \( q_n = x n + \beta + j^{2n} \), log \( \rho_n = -x n/2 + \theta + \zeta(-1)^n \), \( \log \psi_n = 2x n/2 + 2\beta + \theta + \zeta(-1)^n \), while \( b \) and \( c \) are free constants, provides a nonautonomous extension with full parametric freedom. For (2.9) we find the following: log \( q_n = x n + \beta + \zeta(1)^n \), \( \log \rho_n = \theta(-1)^n \), \( \log \psi_n = 2x n + 2\beta + \theta - 2\zeta(-1)^n \) and again \( b \) and \( c \) are free constants. Thus both equations have seven degrees of freedom and we expect the geometry of their transformations to be associated to the aWg E\(_7\).

Next we assume that two terms factor out and we start with the mapping

\[
\frac{(x_n x_{n+1} - q_n q_{n+1})(x_n x_{n-1} - q_n q_{n-1})}{(x_n x_{n+1} - 1)(x_n x_{n-1} - 1)} = \phi_n (x - \psi_n)(x - \omega_n)/(x - \rho_n)(x - \sigma_n) .
\]  

(2.11)

Here is the symmetric form that we obtain

\[
\frac{(x_n x_{n+1} - q_n q_{n+1})(x_n x_{n-1} - q_n q_{n-1})}{(x_n x_{n+1} - 1)(x_n x_{n-1} - 1)} = (x_n - b q_n (x - q_n/b)) 
\]  

(2.12)

and its dual form is

\[
\frac{(x_n x_{n+1} - q_n q_{n+1})(x_n x_{n-1} - q_n q_{n-1})}{(x_n x_{n+1} - 1)(x_n x_{n-1} - 1)} = a_n^2 (x_n - b q_n (x - q_n/b)) .
\]  

(2.13)

The full freedom of (2.12) is encapsulated in the following definition of the parameters: log \( q_n = x n + \beta + j^n + \delta j^{2n} \), log \( \psi_n = x n + \beta + \zeta + j^n + \delta j^{2n} \), log \( \omega_n = x n + \beta - \zeta + j^n + \delta j^{2n} \), with, moreover, \( \phi = 1 \), \( \rho = c \) and \( \sigma = 1/c \) with constant \( c \). The geometry of (2.12) is associated to the aWg D\(_5\). Another integrable symmetric form obtained from (2.11) (and a different singularity pattern) is

\[
\frac{(x_n x_{n+1} - q_n q_{n+1})(x_n x_{n-1} - q_n q_{n-1})}{(x_n x_{n+1} - 1)(x_n x_{n-1} - 1)} = (x_n - b q_n^2 (x_n - q_n^2/b)) 
\]  

(2.14)

and its dual form is

\[
\frac{(x_n x_{n+1} - q_n q_{n+1})(x_n x_{n-1} - q_n q_{n-1})}{(x_n x_{n+1} - 1)(x_n x_{n-1} - 1)} = (x_n - b q_n (x_n - q_n/b)(x_n - 1/c q_n)) .
\]  

(2.15)

The full freedom corresponds to a seven degree of freedom system log \( q_n = x n + \beta + (\gamma + \eta) j^n + (\delta + \theta) j^{2n} \), log \( \psi_n = 2x n + 2\beta + \zeta + j^n + \delta j^{2n} \), log \( \omega_n = 2x n + 2\beta - \zeta + \eta j^n + \theta j^{2n} \), with \( \phi = 1 \), \( \rho = c \), \( \sigma = 1/c \) with constant \( c \) and geometry associated to the aWg E\(_7\).
Finally, we have the symmetric mapping

\[
\frac{(x_n x_{n+1} - g_n g_{n+1})(x_n x_{n-1} - g_n g_{n-1})}{(x_n x_{n+1} - 1)(x_n x_{n-1} - 1)} = \frac{x_n - aq_n^{3/2}}{x_n - aq_n^{-1/2}} = \frac{x_n - bq_n^{3/2}}{x_n - bq_n^{-1/2}}
\]  

(2.16)

with full freedom \( \log q_n = (x + 2\xi(-1)^n)n + \beta \), \( \log \psi_n = (3x/2 + \zeta(-1)^n)n + \beta + \kappa + \eta(-1)^n - \gamma i^n - \delta(-i)^n \), \( \log \omega_n = (3x/2 + \zeta(-1)^n)n + \beta + \theta - \eta(-1)^n + \gamma i^n + \delta(-i)^n \), \( \log \rho_n = (-x/2 + \zeta(-1)^n)n - \beta + \kappa + \eta(-1)^n + \gamma i^n + \delta(-i)^n \). \( \log \sigma_n = (-x/2 + \zeta(-1)^n)n - \beta + \theta - \eta(-1)^n - \gamma i^n - \delta(-i)^n \) and \( \phi = 1 \). Here we again have a seven degrees of freedom system and the geometry of its transformations can be described by the aWg E7.

The last case of degeneracy corresponds to three terms factoring out. In this case we start from the general form

\[
\frac{(x_n x_{n+1} - g_n g_{n+1})(x_n x_{n-1} - g_n g_{n-1})}{(x_n x_{n+1} - 1)(x_n x_{n-1} - 1)} = \phi_n \frac{(x - \psi_n)}{(x - \rho_n)}.
\]  

(2.17)

The first symmetric form we obtain in this case is

\[
\frac{(x_n x_{n+1} - g_n g_{n+1})(x_n x_{n-1} - g_n g_{n-1})}{(x_n x_{n+1} - 1)(x_n x_{n-1} - 1)} = \frac{x_n - aq_n^{3/2}}{x_n - aq_n^{-1/2}}.
\]  

(2.18)

and its dual form is

\[
\frac{(x_n x_{n+1} - g_n g_{n+1})(x_n x_{n-1} - g_n g_{n-1})}{(x_n x_{n+1} - 1)(x_n x_{n-1} - 1)} = \phi_n \frac{x_n - aq_n^{3/2}}{x_n - aq_n^{-1/2}}.
\]  

(2.19)

The full freedom of (2.18), \( \phi = 1 \), corresponds to \( \log q_n = (x + 2\xi(-1)^n)n + \beta \), \( \log \psi_n = (3x/2 + \zeta(-1)^n)n + \beta + \kappa - \gamma i^n - \delta(-i)^n \), \( \log \rho_n = (-x/2 + \zeta(-1)^n)n - \beta + \kappa + \gamma i^n + \delta(-i)^n \) with geometry associated to the aWg D5. The second equation in this family has the symmetric form

\[
\frac{(x_n x_{n+1} - g_n g_{n+1})(x_n x_{n-1} - g_n g_{n-1})}{(x_n x_{n+1} - 1)(x_n x_{n-1} - 1)} = \phi_n \frac{x_n - aq_n^{3/2}}{x_n - aq_n^{-1/2}}.
\]  

(2.20)

In order to obtain the full freedom, we remark that the application of the proper gauge allows to cancel the \((-1)^n\) term in \( \psi_n \) and \( \rho_n \); \( \log q_n = x\eta + \beta + \gamma j^n + \delta j^{2n}, \log \phi_n = x\eta + \eta j^n + \gamma j^n + \delta j^{2n}, \log \psi_n = 3x\eta/2 + \beta + \zeta, \log \rho_n = -x\eta/2 - \beta + \zeta + \gamma j^n + \delta j^{2n} \), corresponding to the aWg D5.

The next equation, which has the symmetric form

\[
\frac{(x_n x_{n+1} - g_n g_{n+1})(x_n x_{n-1} - g_n g_{n-1})}{(x_n x_{n+1} - 1)(x_n x_{n-1} - 1)} = \frac{x_n - g_n^2}{x_n - 1}.
\]  

(2.21)

possesses three more dual forms

\[
\frac{(x_n x_{n+1} - g_n g_{n+1})(x_n x_{n-1} - g_n g_{n-1})}{(x_n x_{n+1} - 1)(x_n x_{n-1} - 1)} = \phi_n \frac{x_n - g_n^2}{x_n - 1}.
\]  

(2.22)
\[
\frac{(x_n x_{n+1} - q_n q_{n+1})(x_n x_{n-1} - q_n q_{n-1})}{(x_n x_{n+1} - 1)(x_n x_{n-1} - 1)} = q_n^2 \frac{x_n - q_n}{x_n - 1/q_n},
\]
(2.23)

\[
\frac{(x_n x_{n+1} - q_n q_{n+1})(x_n x_{n-1} - q_n q_{n-1})}{(x_n x_{n+1} - 1)(x_n x_{n-1} - 1)} = \frac{x_n - q_n}{x_n - 1/q_n}.
\]
(2.24)

The full freedom of (2.21), \( \phi = 1 \) involves the quintic root of unity \( k \) (with \( k^5 = 1 \)).

The net result (with \( \phi = 1 \)) of (2.21) is log \( q_n = zn + \beta - \gamma(k + 1/k)k^n - \delta/k^2 + 2\eta/k^n + \zeta(k + 1/k)k^3n - \eta(k + 1/k)k^4n \), log \( \psi_n = 2zn + 2\beta + \gamma k^3n + \delta k^2n + \zeta k^3n + \eta k^4n \) and \( \rho \) = 1. Again, the geometry of the transformations of this equation is described by the aWg D5.

One more pair of dual mapping is, in symmetric form,

\[
\frac{(x_n x_{n+1} - q_n q_{n+1})(x_n x_{n-1} - q_n q_{n-1})}{(x_n x_{n+1} - 1)(x_n x_{n-1} - 1)} = f q_n \frac{x_n - q_n^2}{x_n - 1},
\]
(2.25)

and

\[
\frac{(x_n x_{n+1} - q_n q_{n+1})(x_n x_{n-1} - q_n q_{n-1})}{(x_n x_{n+1} - 1)(x_n x_{n-1} - 1)} = f q_n \frac{x_n - q_n}{x_n - 1/q_n}.
\]
(2.26)

The full freedom of (2.25) is given by log \( q_n = zn + \beta + \gamma k^n + \delta k^2n + \zeta k^3n + \eta k^4n \), log \( \psi_n = 2zn + 2\beta + 2\gamma(-1)^n \) and \( \rho = 1 \) corresponding again to the aWg D5.

Still another pair of dual mappings exists in symmetric form,

\[
\frac{(x_n x_{n+1} - q_n q_{n+1})(x_n x_{n-1} - q_n q_{n-1})}{(x_n x_{n+1} - 1)(x_n x_{n-1} - 1)} = \frac{x_n - q_n^4}{x - 1},
\]
(2.27)

and

\[
\frac{(x_n x_{n+1} - q_n q_{n+1})(x_n x_{n-1} - q_n q_{n-1})}{(x_n x_{n+1} - 1)(x_n x_{n-1} - 1)} = \frac{x_n - q_n}{x_n - q_n^2}.
\]
(2.28)

For the full freedom of (2.27), we find \( \phi = 1 \), \( \rho = 1 \) and log \( q_n = zn + \beta + \gamma k^n + \delta k^2n + \zeta k^3n + \eta k^4n + \theta k^5n + \kappa^2n \), log \( \psi_n = 4zn + 4\beta + \gamma(k + 2 + 1/k)k^n + \delta(k^2 + 2 + 1/k^2)k^2n + \zeta(k^3 + 2 + 1/k^3)k^3n + \eta(k + 2 + 1/k)k^4n + \theta k^5n + \kappa^2n \). Here the total number of degrees of freedom is 7 and the associated aWg is E7.

The last discrete Painlevé equation in this family is, in symmetric form,

\[
\frac{(x_n x_{n+1} - q_n q_{n+1})(x_n x_{n-1} - q_n q_{n-1})}{(x_n x_{n+1} - 1)(x_n x_{n-1} - 1)} = \frac{x_n - q_n^{5/2}}{x_n - q_n^{3/2}}.
\]
(2.29)

For the full freedom of (2.29) we apply the proper gauge transformation so that no \((-1)^n\) term appears in \( \rho \). The net result (with \( \phi = 1 \)) is log \( q_n = zn + \beta + \gamma(-1)^n + \delta^2 + \zeta^2n \), log \( \psi_n = 5zn/2 + 4\beta + \kappa + \delta^2 + \zeta^2n + \eta^2n + \theta(-1)^n \), log \( \rho_n = -3zn/2 + \kappa + \eta^2 + \theta(-1)^n \). Again the equation has seven degrees of freedom and the symmetry group of its transformations is E7.
3. Limits and degeneracies of the difference Painlevé v equation

This section is devoted to the various integrable mappings related to the d-Pv equation:

\[
\frac{(x_n + x_{n+1} - z_n - z_{n+1})(x_n + x_{n-1} - z_n - z_{n-1})}{(x_n + x_{n+1})(x_n + x_{n-1})} = \frac{(x_n - z_n - a)(x_n - z_n + a)(x_n - z_n - b)(x_n - z_n + b)}{(x_n - c)(x_n + c)(x_n - d)(x_n + d)},
\]

(3.1)

where \(z_n = xn + \beta\) and \(a, b, c, d\) are free constants. One “duality” transformation exists that leaves this equation invariant:

\[
x \to z - x, \quad z \to z,
\]

provided one permutes the numerator and the denominator of the mapping, upon exchange of \(a\) with \(c\) and \(b\) with \(d\).

We start by obtaining the limits of (3.1) when the parameters take some special values. In the process the meaning of \(z\) and of the parameters that survive does not change. By letting \(a \to \infty\) and \(c \to \infty\) simultaneously we find the equation

\[
\frac{(x_n + x_{n+1} - z_n - z_{n+1})(x_n + x_{n-1} - z_n - z_{n-1})}{(x_n + x_{n+1})(x_n + x_{n-1})} = f \frac{(x_n - z_n - b)(x_n - z_n + b)}{(x_n - d)(x_n + d)},
\]

(3.2)

where \(f\) stands for the ratio of \(a^2/c^2\). In this symmetric form, the continuous limit of Eq. (3.2) is a Pv equation with one parameter equal to zero (which makes it a Miura of PIII). Putting \(a = 1 + \varepsilon^2 a_2, \quad b = \varepsilon b_1, \quad d = \varepsilon d_1, \quad z = \sqrt{t}, \quad x = \sqrt{t/(1 - w)}\) we obtain at \(\varepsilon \to 0\) the canonical form of a special case of Pv [13] with \(x = d_1^2/2, \beta = -b_1^2/2, \gamma = a_2^2/2\) and \(\delta = 0\).

The full freedom of (3.2) is encapsulated in the form

\[
\frac{(x_n + x_{n+1} - z_n - z_{n+1})(x_n + x_{n-1} - z_n - z_{n-1})}{(x_n + x_{n+1})(x_n + x_{n-1})} = f \frac{(x_n - z_n + \gamma(-1)^n - b^2)}{x_n^2 - d^2},
\]

(3.3)

a mapping whose geometry is associated to the aWg D4.

Another limit of (3.1) is obtained if we take \(b \to \infty\) and \(f \to 0\) in (3.2), while keeping the product \(g = fb^2\) finite. We obtain the d-Pv

\[
\frac{(x_n + x_{n+1} - z_n - z_{n+1})(x_n + x_{n-1} - z_n - z_{n-1})}{(x_n + x_{n+1})(x_n + x_{n-1})} = \frac{g}{(x_n - d)(x_n + d)}.
\]

(3.4)

Eq. (3.4) has also the dual form
\begin{equation}
\frac{(x_n + x_{n+1} - z_n - z_{n+1})(x_n + x_{n-1} - z_n - z_{n-1})}{(x_n + x_{n+1})(x_n + x_{n-1})} = \frac{(x_n - z_n - b)(x_n - z_n + b)}{g (x_n - c)(x_n + c)}. \tag{3.5}
\end{equation}

The continuous limit of Eq. (3.4) can be shown to be the Painlevé equation P\textsubscript{34}. Indeed, taking $x = e u, g = 1/e^2, d = e^2 \delta$ and $z = 1/e + et$ we obtain, at the limit $e \to 0: u'' = u^2/(2u) - 4u^2 + 4tu - 2\delta^2/u$, which is precisely P\textsubscript{34} in a slightly noncanonical form. As was shown by Nijhoff in Ref. [15], (3.4) is the d-P\textsubscript{34} related to the equation known as the alternate d-P\textsubscript{II}. No extra freedom exists for (3.4) and the geometry of its transformations is described by the aWg B\textsubscript{2}.

We now turn to the study of what we call “degeneracies” of Eq. (3.1), assuming that the numerator and denominator of the mapping have a common factor which drops out. We start by introducing the more general form

\begin{equation}
\frac{(x_n + x_{n+1} - z_n - z_{n+1})(x_n + x_{n-1} - z_n - z_{n-1})}{(x_n + x_{n+1})(x_n + x_{n-1})} = \frac{(x_n - \psi_n)(x_n - z_n - b)(x_n - z_n + b)}{(x_n - \rho_n)(x_n - c)(x_n + c)}. \tag{3.6}
\end{equation}

and apply an integrability criterion in order to single out the possible integrable deautonomisations. The first equation we obtain, in symmetric form, is

\begin{equation}
\frac{(x_n + x_{n+1} - z_n - z_{n+1})(x_n + x_{n-1} - z_n - z_{n-1})}{(x_n + x_{n+1})(x_n + x_{n-1})} = \frac{(x_n - 3z_n/2 + a)}{(x_n + z_n/2 + a)} \left( \frac{(x_n - z_n^2 - b^2)}{x_n^2 - c^2} \right). \tag{3.7}
\end{equation}

Its full freedom corresponds to $z_n = \alpha n + \beta + \gamma j^n + \delta j^{2n}, \rho_n = -\alpha n/2 + \theta + \zeta(-1)^n + \gamma j^n + \delta j^{2n}$ and $\psi_n = 3\alpha n/2 + (\theta + 2\beta) + \zeta(-1)^n$, where $j$ is the cubic root of unity and $b$ and $c$ are constants. We have, thus, a mapping with seven degrees of freedom and geometry associated to the aWg E\textsubscript{7}. A second degeneracy of the same kind leads to the mapping

\begin{equation}
\frac{(x_n + x_{n+1} - z_n - z_{n+1})(x_n + x_{n-1} - z_n - z_{n-1})}{(x_n + x_{n+1})(x_n + x_{n-1})} = \frac{(x_n - 2z)n}{x_n} \frac{(x_n - z_n - b)(x_n - z_n + b)}{(x_n - c)(x_n + c)}. \tag{3.8}
\end{equation}

and its dual form

\begin{equation}
\frac{(x_n + x_{n+1} - z_n - z_{n+1})(x_n + x_{n-1} - z_n - z_{n-1})}{(x_n + x_{n+1})(x_n + x_{n-1})} = \frac{(x_n - z)}{x_n + z} \frac{(x_n - z_n - b)(x_n - z_n + b)}{(x_n - c)(x_n + c)}. \tag{3.9}
\end{equation}

Its full freedom $z_n = \alpha n + \beta + \zeta(-1)^n + \gamma j^n + \delta(-i)^n$, $\rho_n = \theta(-1)^n, \psi_n = 2\alpha n + 2\beta + \gamma j^n + \delta(-i)^n$.
$\theta - 2\zeta(-1)^n$ leads again to a seven degrees of freedom mapping and an $E_7$ description.

At the next level of degeneracy we assume that two common factors drop out and we therefore start with the general form

$$\frac{(x_n + x_{n+1} - z_n - z_{n+1})(x_n + x_{n-1} - z_n - z_{n-1})}{(x_n + x_{n+1})(x_n + x_{n-1})} = \frac{(x_n - \psi_n)(x_n - \omega_n)}{(x - \rho_n)(x - \sigma_n)}, \quad (3.10)$$

where $\psi, \omega, \rho, \sigma$ are, in principle, functions of $n$. The first integrable mapping we obtain is

$$\frac{(x_n + x_{n+1} - z_n - z_{n+1})(x_n + x_{n-1} - z_n - z_{n-1})}{(x_n + x_{n+1})(x_n + x_{n-1})} = \frac{(x_n - z_n - b)(x_n - z_n + b)}{(x_n - c)(x_n + c)}, \quad (3.11)$$

i.e., $\psi_n = z_n + b, \omega_n = z_n - b$ and $\rho = -\sigma = c$, where $b$ and $c$ are constants. The interesting result here is that $z_n$ is an arbitrary function of $n$. Thus (3.11) is not a discrete Painlevé equation but rather a linearisable mapping. This is corroborated by the fact that the growth of the degrees of the iterates of the mapping is linear [16] (rather than quadratic, which characterises the d-$\Psi$s). The linearisation of (3.11) will be presented in the next section. Two more degeneracies of the form (3.10) exist. The first is the equation

$$\frac{(x_n + x_{n+1} - z_n - z_{n+1})(x_n + x_{n-1} - z_n - z_{n-1})}{(x_n + x_{n+1})(x_n + x_{n-1})} = \frac{(x_n - 2z_n - b)(x_n - 2z_n + b)}{(x_n - c)(x_n + c)}, \quad (3.12)$$

and its dual

$$\frac{(x_n + x_{n+1} - z_n - z_{n+1})(x_n + x_{n-1} - z_n - z_{n-1})}{(x_n + x_{n+1})(x_n + x_{n-1})} = \frac{(x_n - z_n - b)(x_n - z_n + b)}{(x_n + z_n - c)(x_n + z_n - c)}. \quad (3.13)$$

The symmetric form of (3.12) can be extended to a seven degree of freedom mapping, $(j^3 = 1)$, $z_n = zn + \beta + (\gamma + \eta)j^n + (\delta + \theta)j^{2n}$, $\psi_n = 2xn + 2\beta + \zeta + j^n + \delta j^{2n}$, $\omega_n = 2xn + 2\beta - \zeta + j^n + \theta j^{2n}$ and $\rho = -\sigma = c$, where $c$ is a free constant, with geometry described by the aWg $E_7$.

The second degeneracy of (3.10) type is, in symmetric form,

$$\frac{(x_n + x_{n+1} - z_n - z_{n+1})(x_n + x_{n-1} - z_n - z_{n-1})}{(x_n + x_{n+1})(x_n + x_{n-1})} = \frac{x_n - 3z_n/2 - a x_n - 3z_n/2 - b}{x_n + z_n/2 - a x_n + z_n/2 - b}. \quad (3.14)$$

As is the previous case the full freedom $z_n = (x + 2\zeta(-1)^n)n + \beta$, $\psi_n = (3\zeta/2 + \zeta(-1)^n)n + 3\beta/2 + \kappa + \eta(-1)^n - \gamma j^n - \delta(-1)^n$, $\omega_n = (3\zeta/2 + \zeta(-1)^n)n + 3\beta/2 + \theta - \gamma j^n - \delta(-1)^n$. 

\[ \eta(-1)^n + \gamma i^n + \delta(-i)^n, \quad \rho_n = (-\alpha/2 + \zeta(-1)^n)n - \beta/2 + \kappa + \eta(-1)^n + \gamma i^n + \delta(-i)^n, \]
\[ \sigma_n = (-\alpha/2 + \zeta(-1)^n)n - \beta/2 + \theta - \eta(-1)^n - \gamma i^n - \delta(-i)^n \]
again leads to a mapping described by Eq. 7.

The last case of degeneracy occurs when all but one factor in the numerator and denominator of the rhs of the equation drop out. In order to investigate this case we introduce the general form

\[ \frac{(x_n + x_{n+1} - z_n - z_{n+1})(x_n + x_{n-1} - z_n - z_{n-1})}{(x_n + x_{n+1})(x_n + x_{n-1})} = f \frac{x_n - 3z_n/2 - a}{x_n + z_n/2 - a}, \]

where, in principle, \( \phi, \psi, \rho \) may depend on \( n \). The first integrable case we obtain can be written in symmetric form as

\[ \frac{(x_n + x_{n+1} - z_n - z_{n+1})(x_n + x_{n-1} - z_n - z_{n-1})}{(x_n + x_{n+1})(x_n + x_{n-1})} = f \frac{x_n - 3z_n/2 - a}{x_n + z_n/2 - a}, \]

where \( f = f \) is a free constant. In order to obtain the full freedom of (3.16), we apply the adequate gauge so that no \((-1)^n\) term appears in \( \psi \) and \( \rho \). We find \( z_n = \alpha n + \beta + \gamma i^n + \delta(-i)^n \), \( \psi_n = 3\alpha n/2 + 3\beta/2 + \zeta, \rho_n = -\alpha n/2 - \beta/2 + \zeta + \gamma i^n + \delta(-i)^n \). The geometry of this equation is related to the \( aW^g D4 \). The next equation in the same family is, in symmetric form,

\[ \frac{(x_n + x_{n+1} - z_n - z_{n+1})(x_n + x_{n-1} - z_n - z_{n-1})}{(x_n + x_{n+1})(x_n + x_{n-1})} = f \frac{x_n - 3z_n/2 - a}{x_n + z_n/2 - a}. \]

It turns out that the full freedom of this mapping involves a free function \( h(n) \). We have \( (\phi = 1) z_n = -h(n + 1) - h(n - 1) + 2a, \psi_n = -h(n + 1) - h(n) - h(n - 1) + 4a, \rho_n = h(n) \) and the presence of this arbitrary function signals a linearisable equation. We shall show indeed, in Section 4, how this mapping can be linearised.

The next case is

\[ \frac{(x_n + x_{n+1} - z_n - z_{n+1})(x_n + x_{n-1} - z_n - z_{n-1})}{(x_n + x_{n+1})(x_n + x_{n-1})} = f \frac{x_n - 2z_n}{x_n}, \]

and its dual form

\[ \frac{(x_n + x_{n+1} - z_n - z_{n+1})(x_n + x_{n-1} - z_n - z_{n-1})}{(x_n + x_{n+1})(x_n + x_{n-1})} = f \frac{x_n - z_n}{x_n + z_n}, \]

where \( f \) is a free constant. Investigating the full freedom of (3.18) we find \( (\rho = 0) z_n = \alpha n + \beta + \delta(-i)^n \) and \( \psi_n = 2\alpha n + 2\beta + 2\gamma(-1)^n \). The geometry of the transformations of this d-\( P \) is described by the \( aW^g D4 \).

Another linearisable mapping can be obtained from (3.15). In symmetric form we find

\[ \frac{(x_n + x_{n+1} - z_n - z_{n+1})(x_n + x_{n-1} - z_n - z_{n-1})}{(x_n + x_{n+1})(x_n + x_{n-1})} = f \frac{x_n - 2z_n}{x_n}, \]

and its dual

\[ \frac{(x_n + x_{n+1} - z_n - z_{n+1})(x_n + x_{n-1} - z_n - z_{n-1})}{(x_n + x_{n+1})(x_n + x_{n-1})} = f \frac{x_n - z_n}{x_n + z_n}. \]
The full freedom of (3.20) involves a free function $h(n)$ and we have ($\phi = 1$, $\rho = 0$) $z_n = h(n + 1) + h(n - 2)$. $\psi_n = h(n + 1) + h(n) + h(n - 1) + h(n - 2)$. Again, the linearisation of (3.20) will be presented in the next section.

The remaining integrable mappings obtained from (3.15) are of d-PV type. The first case is

$$\frac{(x_n + x_{n+1} - z_n - z_{n+1})(x_n + x_{n-1} - z_n - z_{n-1})}{(x_n + x_{n+1})(x_n + x_{n-1})} = \frac{x_n - 4z_n}{x_n} \quad (3.22)$$

with dual form

$$\frac{(x_n + x_{n+1} - z_n - z_{n+1})(x_n + x_{n-1} - z_n - z_{n-1})}{(x_n + x_{n+1})(x_n + x_{n-1})} = \frac{x - z}{x - 3z} \quad (3.23)$$

The symmetric form (3.22) can be extended ($\phi = 1$, $\rho = 0$) to a form involving the quintic root of unity $k$. We find $z_n = zn + n + \gamma k^n + \delta k^{2n} + \xi k^{3n} + \eta k^{4n} + \theta j^n + \kappa j^{2n}$ and

$$\psi_n = 4xn + b + \gamma (k + 2 + 1/k)k^n + \delta (k^2 + 2 + 1/k^2)k^{2n} + \xi (k^2 + 2 + 1/k^2)k^{3n} + \eta (k + 2 + 1/k)k^{4n} + \theta j^n + \kappa j^{2n}.$$ 

Thus (3.22) has seven degrees of freedom and its geometry is related to the aWg E7.

The last degeneracy obtained from (3.15) is

$$\frac{(x_n + x_{n+1} - z_n - z_{n+1})(x_n + x_{n-1} - z_n - z_{n-1})}{(x_n + x_{n+1})(x_n + x_{n-1})} = \frac{x_n - a - 5zn/2}{x_n - a + 3zn/2} \quad (3.24)$$

In order to recover the full freedom of (3.24) we first introduce a gauge such that no $(-1)^n$ term appears in $\rho$ and find ($\phi = 1$): $z_n = zn + n + \gamma (-1)^n + \delta j^n + \xi j^{2n}$, $\psi_n = 5zn/2 + 4b + \kappa + \delta j^n + \xi j^{2n} + \eta j^n + \theta (-i)j^n$ and $\rho_n = -3zn/2 + \kappa + \eta j^n + \theta (-i)j^n.$

Again we have a seven degrees of freedom equation the geometry of which is described by the aWg E7.

4. Linearising the linearisable mappings

In the previous section we have investigated integrable mappings obtained as equations that have the d-PV general form. Among all the possible integrable cases, we found three cases for which the application of the degree growth criterion indicated that they should be integrable by linearisation. Such an occurrence is not without precedent. Indeed when we carried out a similar analysis for equations derived from d-PIV we obtained the mapping $(x_n + x_{n+1})(x_n + x_{n-1}) = h(n)(x_n^2 - 1)$, where $h(n)$ is a free function of $n$. As was shown in Ref. [1] this mapping is a special case of the discrete Gambier equation [17].

In this section we will proceed to the linearisation of the three mappings obtained in Section 3. The first mapping (3.11) has the form

$$\frac{(x_n + x_{n+1} - z_n - z_{n+1})(x_n + x_{n-1} - z_n - z_{n-1})}{(x_n + x_{n+1})(x_n + x_{n-1})} = \frac{(x_n - z_n)^2 - b^2}{x_n^2 - c^2} \quad (4.1)$$

where $z_n$ is an arbitrary function of $n$. The method for the integration of (4.1) was already given in [12]. We start by rewriting it in a more convenient form as
where \( m \) and \( k \) are constants. Next we compute the discrete derivative of (4.2) obtained by using the fact that \( k \) is a constant, i.e., eliminating \( k \) between (4.2) and its up-shift, resulting to a four-point equation. Next we write the linear equation obtained by using the fact that \( \frac{\partial(z_n)}{\partial(z_{n+1})} \) using the fact that \( \frac{\partial(z_n)}{\partial(z_{n-1})} \) is a constant, i.e., eliminating \( k \) between (4.3) and its up-shift, the four-point equation thus obtained is identical to the one we found starting from (4.2). The way to integrate the latter is to start from two initial conditions for \( x \), say \( x_{-1} \) and \( x_0 \) and two constants \( m, k \). Use (4.2) to compute \( x_1 \), substitute all three values of \( x \) into (4.3) and fix \( \kappa \) so that the latter is satisfied. From then on, integrate the linear equation (4.3) for this value of \( \kappa \) to obtain all the \( x \)’s.

The second mapping (3.17) to linearise is of the form

\[
\frac{(x_n + x_{n+1} - z_n - z_{n+1})(x_n + x_{n-1} - z_n - z_{n-1})}{(x_n + x_{n+1})(x_n + x_{n-1})} = \frac{(x_n - z_n)^2 + m + k}{x_n^2 - m + k},
\]

(4.2)

where \( m \) and \( k \) are constants. Next we compute the discrete derivative of (4.2) obtained by using the fact that \( k \) is a constant, i.e., eliminating \( k \) between (4.2) and its up-shift, resulting to a four-point equation. Next we write the linear equation

\[
z_n x_n = m - \kappa + \left( \kappa + \frac{z_n^2}{2} \right) \left( \frac{x_n + x_{n+1}}{z_n + z_{n+1}} + \frac{x_n + x_{n-1}}{z_n + z_{n-1}} \right),
\]

(4.3)

where a free constant \( \kappa \) appears. It turns out that if we compute the discrete derivative of (4.3) using the fact that \( \kappa \) is a constant (i.e., eliminating \( \kappa \) between (4.3) and its up-shift), the four-point equation thus obtained is identical to the one we found starting from (4.2). The way to integrate the latter is to start from two initial conditions for \( x \), say \( x_{-1} \) and \( x_0 \) and two constants \( m, k \). Use (4.2) to compute \( x_1 \), substitute all three values of \( x \) into (4.3) and fix \( \kappa \) so that the latter is satisfied. From then on, integrate the linear equation (4.3) for this value of \( \kappa \) to obtain all the \( x \)’s.

The second mapping (3.17) to linearise is of the form

\[
\frac{(x_n + x_{n+1} - z_n - z_{n+1})(x_n + x_{n-1} - z_n - z_{n-1})}{(x_n + x_{n+1})(x_n + x_{n-1})} = \frac{x_n - \psi_n}{x_n - \rho_n},
\]

(4.4)

where \( \rho_n \) is now a free function \( h(n) \) and \( z_n = -h(n+1) - h(n-1) + a, \ \psi_n = -h(n+1) - h(n) - h(n-1) + 2a \). We start by introducing an auxiliary variable

\[
y_n = -\frac{x_n + x_{n-1} - z_n - z_{n-1}}{x_n + x_{n-1}},
\]

(4.5)

whereupon (4.4) can be rewritten as

\[
y_n y_{n+1} - 1 = \frac{\rho_n - \psi_n}{x_n - \rho_n}.
\]

(4.6)

Next we construct the following quantity

\[
\frac{2\xi_{n+1/2}}{y_n y_{n+1} - 1} + \frac{2\xi_{n-1/2}}{y_{n-1} y_n - 1} = x_n + x_{n-1} - \rho_n - \rho_{n-1},
\]

(4.7)

where \( 2\xi_{n+1/2} = \rho_n - \psi_n = 2\rho_n + \rho_{n+1} + \rho_{n-1} - 2a \). Using (4.5) we can recast the r.h.s. of (4.7) into the form

\[
x_n + x_{n-1} - \rho_n - \rho_{n-1} = -\left( \frac{\xi + \xi_{n+1/2} + \xi_{n-1/2}}{1 + y_n Z_n} \right).
\]

(4.8)

We find that \( \alpha = a, \ 2Z_n = -2a + \rho_n + \rho_{n+1} + \rho_{n-1} + \rho_{n-2} \) and we have precisely \( Z_n + Z_{n+1} = \xi_{n+1/2} + \xi_{n+3/2} \). Thus mapping (4.4) is reduced exactly to a mapping whose linearisation was examined in Ref. [12].

We turn now to the third linearisable case (3.20)

\[
\frac{(x_n + x_{n+1} - z_n - z_{n+1})(x_n + x_{n-1} - z_n - z_{n-1})}{(x_n + x_{n+1})(x_n + x_{n-1})} = \frac{x_n - \psi_n}{x_n},
\]

(4.9)
where \( z \) and \( c \) are expressed in terms of a free function \( h(n) \) as \( \psi_n = h(n+1) + h(n) + h(n-1) + h(n-2) \), \( z_n = h(n+1) + h(n-2) \). In order to linearise this mapping we start by multiplying the free function \( h \) by some constant \( k \) and eliminate \( k \) between (4.9) and its up-shift. In the resulting equation, which is homogeneous in \( x \), we introduce the following substitution:

\[
\frac{x_{n+1}}{x_n} = \frac{\varphi_n}{y_n} - 1,
\]

where \( \varphi_n = h(n+2) + h(n+1) + h(n-1) + h(n-2) \). We obtain thus a second-order mapping for \( y \). It turns out that this mapping can be expressed as a discrete derivative of the homographic mapping \( H_n = y_{n-1}y_n - u_ny_{n-1} - v_ny_n + w_n \) where \( u_n = h(n-2) + h(n-1) + h(n) + h(n+1) \), \( v_n = h(n-1) - h(n+1) \) and \( w_n = h(n-2)h(n-1) - h(n-2)h(n+1) + h(n-1)^2 + h(n-1)h(n) + h(n+1)h(n) + h(n)^2 \). This means that we can start from the equation

\[
y_{n-1}y_n - u_ny_{n-1} - v_ny_n + w_n = m,
\]

where \( m \) is a constant, and eliminate \( m \) between (4.11) and its up-shift in order to obtain the mapping for \( y \) resulting from (4.9). The integration method for (4.9) now becomes obvious.

Starting from initial conditions for \( x \), say \( x_0 \) and \( x_1 \), we can use (4.9) to obtain \( x_2 \). Then using (4.10) we find \( y_0 \), \( y_1 \). Substituting into (4.11) we can obtain the value of \( m \) and, once we have its value, we can integrate this homographic mapping (through linearisation) and get \( y_n \) for all values of \( n \). Having \( y_n \) it is straightforward to compute \( x \) from (4.10) by solving another linear equation.

5. Conclusion

In this paper we have presented the various integrable equations which can be obtained from the d-P\(_V\) and q-P\(_{VI}\) following the procedures introduced in CF. Given the rich form of these equations it was not astonishing that we were able to obtain a host of new integrable mappings (and, as a matter of fact, more limits and/or degeneracies could possibly exist, having eluded the present analysis). This paper complements the results of CF, which was limited by the fact that the forms of d-P\(_V\) and q-P\(_{VI}\) were not known at that time.

The various equations obtained here can be classified in two broad classes: discrete Painlevé equations and linearisable mappings. For the first we have obtained the maximal number of parameters, using a discrete integrability criterion. We have shown that, except in the case of some equations obtained through limits, all equations have a number of parameters equal or superior to that of Painlevé VI. In the case of the linearisable equations we have presented their solution either by reducing them to mappings that we have integrated in previous publications or by explicitly constructing their effective linearisation.
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