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Dimensionality effects in the ideal Bose and Fermi gases

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A recently established equivalence between the ideal Bose and Fermi gases [M. H. Lee, Phys. Rev. E **55**, 1518 (1997)] is further analyzed with emphasis on the dimensionality effects. The equivalence is shown to be a peculiarity of the bosonlike correlations in two dimensions. [S1063-651X(97)03710-0]

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Recently [1], Lee established a remarkable equivalence between the ideal Bose and Fermi gases in two dimensions. The equivalence is based on a certain invariance of the polylogarithms [2] (see also Ref. [3]) under Euler's transform of the fugacities of the two gases, an invariance found many years ago by Landen [4]. The result might have been expected since a nonrelativistic two-dimensional gas is equivalent to a relativistic gas in one dimension. We further analyze in this paper the underlying aspects of this equivalence with respect to the dimensionality effects and show that the reason for this remarkable particularity in two dimensions resides in the combined effect of the bosonlike correlations (responsible for the Bose-Einstein condensation) and dimensionality.

We begin with a brief review of Lee's result. The basic object is the number of ''thermal states''

$$\nu=\frac{N\lambda^2}{gA}, \tag{1}$$

where N is the number of particles, A is the area occupied by the gas, g is a kinematic factor of degeneracy, and

$$\lambda=\left(\frac{2\pi\hbar^2}{mT}\right)^{1/2} \tag{2}$$

is the thermal wavelength; in Eq. (2) $\hbar$ is Planck's constant, m is the mass of a particle, and T ($=1/\beta$) is the temperature. Introducing the inter-particle spacing $a=(A/N)^{1/2}$ and the characteristic energy $\varepsilon_0=2\pi\hbar^2/gma^2$, we get $\nu=\varepsilon_0/T$, which justifies the designation ''number of thermal states''; for a Fermi gas $\varepsilon_0=2\varepsilon_F$, where ε_F is the Fermi energy. The number of thermal states is given by $\nu_b=-\ln(1-z_b)$ for bosons and $\nu_f=\ln(1+z_f)$ for fermions, where $z_{b,f}$ are the fugacities. For $\nu_b=\nu_f=\nu$ we get $1+z_f=(1-z_b)^{-1}$, which is precisely Euler's transform between z_b and $-z_f$. We note that $z_b=1-\exp(-\nu_b)$ and $z_f=\exp(\nu_f)-1$. The energies of the two gases are given by

$$\beta\nu_b E_b/N_b=\mathrm{Li}_2(z_b),$$

$$\beta\nu_f E_f/N_f=-\mathrm{Li}_2(-z_f), \tag{3}$$

where $\mathrm{Li}_2(z)$ is the dilog of z. A useful integral representation of the polylogs is [2]

$$\mathrm{Li}_{n+1}(z)=\frac{1}{\Gamma(n+1)}\int_0^z du\left(\ln\frac{z}{u}\right)^n\frac{1}{1-u} \tag{4}$$

for $\mathrm{Re}z<1$. Under Euler's transform between z_b and $-z_f$ given above, the dilog becomes

$$\mathrm{Li}_2(z_b)=-\mathrm{Li}_2(-z_f)-\frac{1}{2}\mathrm{Li}_1^2(-z_f), \tag{5}$$

which is precisely Landen's relation [1,4]. Using Eq. (5), we obtain straightforwardly

$$E_b/N_b=E_f/N_f-\frac{1}{2}\varepsilon_F, \tag{6}$$

which is Lee's main result. In addition, it follows from (6) that the specific heats of the two gases are equal, a result previously established [5]. Since $\Omega=-E$ in two dimensions [6], where the thermodynamic potential $\Omega=-(1/\beta)\ln Q=-pA$, Q being the grand-partition function and p being the pressure, we have also

$$p_b/n_b=p_f/n_f-\frac{1}{2}\varepsilon_F \tag{7}$$

and the equality of the entropies $S_b/N_b=S_f/N_f$. These relations establish a perfect equivalence between the ideal Bose and Fermi gases in two dimensions.

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We now investigate the possibility of such an equivalence in one or three dimensions. Naturally, we shall be interested in high values of the number of thermal states ν, $\nu \gg 1$, i.e., in temperatures much lower than the degeneracy temperature. In three dimensions we have $\nu = (\varepsilon_0/T)^{3/2}$ and $\varepsilon_0 = (4/9)^{1/3}\varepsilon_F$ for the Fermi gas. For an ideal Fermi gas the number of thermal states is given by

$$\nu_f = -\Gamma(3/2)\mathrm{Li}_{3/2}(-z_f), \tag{8}$$

and making use of the well-known integrals with the Fermi-Dirac distribution we get [7]

$$\nu_f = \frac{2}{3}(\ln z_f)^{3/2}\left[1 + \frac{\pi^2}{8}\frac{1}{(\ln z_f)^2} + \cdots\right]; \tag{9}$$

we see that $z_f \gg 1$ for $\nu_f \gg 1$. Similarly, for an ideal Bose gas we have

$$\nu_b = \Gamma(3/2)\mathrm{Li}_{3/2}(z_b). \tag{10}$$

However, in contrast to the two-dimensional case, an interesting phenomenon occurs in three dimensions as a result of the bosonlike correlations, a phenomenon that is in fact the Bose-Einstein condensation [8]. Indeed, a simple change of variable in Eq. (4) leads to

$$\mathrm{Li}_{3/2}(z_b) = \sum_{n=1}^{\infty} \frac{z_b^n}{n^{3/2}}. \tag{11}$$

For $0 < z_b < 1$ this series is bounded by $\mathrm{Li}_{3/2}(1) = \zeta(3/2)$, where ζ is Riemann's zeta function. Therefore, ν_b is bounded by $\Gamma(3/2)\zeta(3/2)$, which means that the bosons condense on the zero-energy level. Consequently, ν_b cannot be equal to ν_f and the two gases are not equivalent.

A similar situation appears in one dimension, though not for the number of thermal states, but for the energy. The number of thermal states in one dimension is $\nu = (\varepsilon_0/T)^{1/2}$, where $\varepsilon_0 = 4\varepsilon_F$ for the Fermi gas. For an ideal Fermi gas in one dimension we have

$$\nu_f = -\Gamma(1/2)\mathrm{Li}_{1/2}(-z_f) \tag{12}$$

and

$$\nu_f = 2(\ln z_f)^{1/2}\left[1 - \frac{\pi^2}{24}\frac{1}{(\ln z_f)^2} + \cdots\right] \tag{13}$$

in the asymptotic regime $\nu_f, z_f \gg 1$. Similarly the energy is given by

$$\begin{aligned}\beta\nu_f E_f/N_f &= -\Gamma(3/2)\mathrm{Li}_{3/2}(-z_f)\\ &= \frac{2}{3}(\ln z_f)^{3/2}\left[1 + \frac{\pi^2}{8}\frac{1}{(\ln z_f)^2} + \cdots\right],\end{aligned} \tag{14}$$

whence

$$E_f/N_f = \frac{1}{3}\varepsilon_F\left(1 + \frac{\pi^2}{4}\frac{T^2}{\varepsilon_F^2} + \cdots\right). \tag{15}$$

From Eq. (15) we obtain the well-known specific heat of an ideal Fermi gas

$$c_f = \frac{\pi^2}{6}\frac{T}{\varepsilon_F}. \tag{16}$$

For an ideal Bose gas in one dimension we have

$$\nu_b = \Gamma(1/2)\mathrm{Li}_{1/2}(z_b) = \sqrt{\pi}\sum_{n=1}^{\infty}\frac{z_b^n}{\sqrt{n}} \tag{17}$$

and the series given by Eq. (17) diverges for $z_b \to 1$. Therefore, we could have a relationship between z_b and z_f for $\nu_b = \nu_f = \nu \gg 1$. However, the energy of the Bose gas is given by

$$\beta\nu_b E_b/N_b = \Gamma(3/2)\mathrm{Li}_{3/2}(z_b) = \Gamma(3/2)\sum_{n=1}^{\infty}\frac{z_b^n}{n^{3/2}} \tag{18}$$

and we see again that the energy per particle is now bounded, in contrast to the Fermi case. Moreover, for $z_b \to 1$ we get

$$E_b/N_b \cong \frac{1}{4}\sqrt{\frac{\pi}{\varepsilon_F}}\zeta(3/2)T^{3/2}; \tag{19}$$

comparing this equation with Eq. (15), we see that there can be no equivalence in one dimension of the type established by Lee in two dimensions.

The arguments presented above can be summarized as follows. The number of thermal states for fermions in dimension d is given by

$$\nu_f^{(d)} \sim \int_0^{\infty} dx\, x^{d/2-1}\frac{z_f}{e^x + z_f}, \tag{20}$$

and in the asymptotic regime $\nu_f \gg 1$, $z_f \to \infty$, we obtain

$$\nu_f^{(d)} \sim (\ln z_f)^{d/2}. \tag{21}$$

For an ideal gas of bosons $\nu_b \gg 1$ corresponds to $z_b \to 1$ and the bosonlike correlations expressed by the singularity of the Bose-Einstein distribution at vanishing energy determine distinct asymptotic behaviors of ν_b with dimension d. In three dimensions $\nu_b^{(3)}$ is finite for $z_b \to 1$, indicating Bose-Einstein condensation; thus there cannot be any equivalence between bosons and fermions in this case. In one dimension $\nu_b^{(1)}$ diverges for $z_b \to 1$, like $\nu_b^{(1)} \sim (1 - z_b)^{-1/2}$, and a relationship with $\nu_f^{(1)}$ given by Eq. (21) might be possible; however, the energy per particle in this case goes like $E_b/N_b \sim T^{3/2}$ for bosons, while $E_f/N_f \sim \text{const} + T^2$ for fermions, according to Eqs. (15) and (19), and we see again that there could not be any equivalence. In two dimensions $\nu_b^{(2)} = -\ln(1-z_b)$; comparing this equation with Eq. (21), we see that such an equivalence might be possible via Euler's transform between the two fugacities; in addition, $E_b/N_b \sim T^2$ and $E_f/N_f \sim \text{const} + T^2$ in this case, which makes this equivalence even more likely. However, its precise demonstration, as given by Lee [1], remains a remarkable property of the ideal Bose and Fermi gases in two dimensions.

[1] M. H. Lee, Phys. Rev. E **55**, 1518 (1997).

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[4] L. Lewin, *Dilogarithms and Associated Functions* (McDonald, London, 1958).

[5] R. M. May, Phys. Rev. **135**, A1515 (1964); J. G. Dash, *Films on Solid Surfaces* (Academic, New York, 1975).

[6] Making use of Eq. (3) and expressing the potential Ω directly with the dilog, we obtain also the addition formula $\mathrm{Li}_2(1-z)+\mathrm{Li}_2(z)=\mathrm{Li}_2(1)-\mathrm{Li}_1(z)\mathrm{Li}_1(1-z)$.

[7] By the same method we can obtain the asymptotic formula

$$\mathrm{Li}_{n+1}(-z) = -\frac{1}{(n+1)\Gamma(n+1)}(\ln z)^{n+1} \times\left[1+\frac{\pi^2}{6}n(n+1)\frac{1}{(\ln z)^2}+\cdots\right]$$

for $z \gg 1$

[8] As is well known, there is no superfluid transition in two dimensions; see, for example, M. F. M. Osborne, Phys. Rev. **76**, 396 (1949).