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Displaced logarithmic profile of the velocity distribution in the boundary layer of a turbulent flow over an unbounded flat surface

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ABSTRACT

It is shown that the Reynolds equations for a turbulent flow over an unbounded flat surface in the presence of a constant pressure-gradient lead to a displaced logarithmic profile of the velocity distribution; the displaced logarithmic profile is obtained by assuming a constant production rate of turbulence energy. The displacement height measured on the (vertical) axis perpendicular to the surface is either positive or negative. For a positive displacement height the boundary layer exhibits an inversion, while for a negative displacement height the boundary layer is a direct one. In an inversion boundary layer the logarithmic velocity profile is disrupted into two distinct branches separated by a logarithmic singularity. The viscosity transforms this logarithmic singularity into a sharp edge, governed by a generalized Reynolds number. The associated temperature distribution is calculated, and the results are discussed in relation to meteorological boundary-layer jets and stratified layers. The effects of gravitation and atmospheric thermal or fluid-mixture concentration gradients ("external forcings") are also considered; it is shown that such circumstances may lead to various modifications of the boundary layers. A brief presentation of a similar situation is described for a circular pipe.

Usually, the velocity logarithmic profile in the boundary layer of a turbulent flow over an unbounded flat surface is derived by using dimensional or similarity arguments [1,2]. We examine here the implications of the Reynolds equations for the turbulent boundary layer in the presence of a pressure gradient. It is shown that a constant pressure-gradient leads to a linear dependence of the Reynolds shear stress on the distance from the surface; such a dependence, combined with the assumption of a constant production rate of turbulence energy, yields a displaced logarithmic profile of velocity. The displacement height, measured along the (vertical) axis perpendicular to the surface, is either positive or negative, corresponding to an inversion or a direct boundary layer, respectively. The difference between the two types of boundary layers arises from the boundary conditions. In the inversion layer the fluid flows in the direction opposite to the main flow, and the logarithmic law of the velocity profile is splitted into two branches separated by a logarithmic singularity. The viscosity transforms this singularity into a sharp edge, governed by a generalized Reynolds number. The temperature distribution associated with such boundary layers is calculated, and the results are discussed in connection with the meteorological boundary-layer jets and stratified layers. Gravitation and atmospheric thermal or fluid-mixture concentration gradients ("external forcings") are also considered; it is shown that such circumstances may lead to various modifications of the boundary layers. A similar situation is presented for a circular pipe.

Specifically, we are interested in a turbulent flow of an incompressible fluid along an infinite plane surface. The coordinates x and y lie on the surface and the coordinate z is perpendicular to the surface (vertical coordinate). The velocity components (u, v, w) correspond to the (x, y, z)-directions. The fluid flows along the x axis with velocity u. As usually, we introduce the mean velocities $\bar{u}$, $\bar{v}$ and $\overline{w}$ and the fluctuating velocities u', v' and w', by $u \to \bar{u} + u'$, etc. (we consider time averaging; also, spatial averaging will be discussed below). We assume $\bar{v} = \overline{w} = 0$ and $\bar{u}(z) \neq 0$ depending only on z (a uniform flow along the x-direction). Under these conditions the Navier–Stokes equations lead to the Reynolds equations [1,3,4]

$$\begin{aligned}
0 &= -\frac{1}{\rho}\frac{\partial p}{\partial x} - \frac{\partial}{\partial x}\overline{u'^2} - \frac{\partial}{\partial y}\overline{u'v'} - \frac{\partial}{\partial z}\overline{u'w'} + \nu\frac{\partial^2 \bar{u}}{\partial z^2} , \\
0 &= -\frac{\partial}{\partial x}\overline{u'v'} - \frac{\partial}{\partial y}\overline{v'^2} - \frac{\partial}{\partial z}\overline{v'w'} , \\
0 &= -\frac{\partial}{\partial x}\overline{u'w'} - \frac{\partial}{\partial y}\overline{v'w'} - \frac{\partial}{\partial z}\overline{w'^2} ,
\end{aligned} \tag{1}$$

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where ρ is the fluid density, p is the (mean) pressure (depending only on x) and ν is the viscosity coefficient. We note the occurrence in equation (1) of the correlation functions $\overline{u'^2}$, $\overline{u'v'}$, etc. (also called variances, like $\overline{u'^2}$, and covariances, like $\overline{u'v'}$); these quantities are the components of the Reynolds stress tensor (which, multiplied by ρ, is a momentum flux density). We assume a constant, negative pressure-gradient $\partial p/\partial x = const < 0$. We note that the Reynolds stress tensor generates forces which may compete with the pressure-gradient force in equation (1) (and with the viscosity "force"); therefore, equation (1) is in fact an equilibrium equation (corresponding to a steady flow), as expected.

Equations (1) represent a system of three equations with seven unknowns: the components of the Reynolds tensor and the mean velocity $\bar{u}$; it is an under-determined system of equations. We are interested in the first equation (1), where we assume $\partial(\overline{u'v'})/\partial y = 0$, $\overline{u'w'}(z) \neq 0$ depending only on z and $\partial\overline{u'^2}/\partial x = const$. In general, constant Reynolds stress components with respect to a coordinate amount to a homogeneous turbulence along that axis [5–7]. Under these conditions, the first equation (1) reads

$$0 = A - \frac{d}{dz}\overline{u'w'} + \nu\frac{d^2\bar{u}}{dz^2} \ , \tag{2}$$

where $A = -(1/\rho)dp/dx - \partial\overline{u'^2}/\partial x$; for the sake of generality we keep for the moment $\partial\overline{u'^2}/\partial x = const$ in equation (2), corresponding to an inhomogeneous turbulence along the x-axis.

We leave aside for the moment the viscosity term in equation (2); then, the integration of this equation gives

$$\overline{u'w'} = Az - \beta u_*^2 \ , \tag{3}$$

where $\beta = \pm 1$ and the parameter u_* is a surface friction velocity; for the sake of generality we keep both signs in the boundary condition $\overline{u'w'}\mid_{z=0} = -\beta u_*^2$; ρu_*^2 is the friction force per unit area of the surface and $\rho\overline{u'w'}$ is the xz-component of the momentum flux density (Reynolds shear stress). Equation (3) can also be written as

$$\overline{u'w'} = \frac{\beta u_*^2}{h}(z-h) \ , \ h = \frac{\beta u_*^2}{A} \ , \ A \neq 0. \tag{4}$$

Such a linear dependence of the shear stress is known in the atmospheric turbulence of the boundary layers [8] and in turbulent flow on flat plates or in channels [1]. We note that the displacement height h may have both signs.

Multiplying the Navier–Stokes equations by $\bar{u}$ and using the same procedure ($u \to \bar{u} + u'$, etc.), we get the conservation law for the mean-flow energy

$$0 = \overline{\frac{\partial}{\partial t}\left[\frac{1}{2}\bar{u}^2\right]} = \\ = \bar{u}\left[-\frac{1}{\rho}\frac{\partial p}{\partial x} - \frac{\partial}{\partial x}\overline{u'^2} - \frac{\partial}{\partial y}\overline{u'v'} - \frac{\partial}{\partial z}\overline{u'w'} + \nu\frac{\partial^2\bar{u}}{\partial z^2}\right] \ , \tag{5}$$

which is the first equation (1) multiplied by $\bar{u}$. Similarly, multiplying the Navier–Stokes equations by the fluctuating velocities and taking the average we get the conservation equation for the turbulence energy

$$\begin{aligned} 0 \ &= \ \overline{\tfrac{\partial}{\partial t}\left[\tfrac{1}{2}(u'^2 + v'^2 + w'^2)\right]} = \\ &= -\overline{u'w'}\tfrac{\partial\bar{u}}{\partial z} - \tfrac{1}{2}\bar{u}\tfrac{\partial}{\partial x}\left(\overline{u'^2} + \overline{v'^2} + \overline{w'^2}\right) + \\ &\quad + \nu\left(\overline{u'\Delta u'} + \overline{v'\Delta v'} + \overline{w'\Delta w'}\right) \ , \end{aligned} \tag{6}$$

where third-order terms involving products of three fluctuating velocities and velocity derivatives, as well as the contribution of the fluctuating part of the pressure have been dropped out; in addition, in deriving equation (6) the continuity equation $\partial u'/\partial x + \partial v'/\partial y + \partial w'/\partial z = 0$ has been used. The main assumption made here is that the fluctuations are small in comparison with the mean flow. Adding the two equations (5) and (6), we get the conservation law of the total energy

$$\begin{aligned} 0 \ &= \ \overline{\tfrac{\partial}{\partial t}\left[\tfrac{1}{2}(u^2 + v^2 + w^2)\right]} = \\ &= -\tfrac{1}{\rho}\tfrac{\partial p}{\partial x}\bar{u} - \tfrac{\partial}{\partial x}\left(\bar{u}\overline{u'^2}\right) - \tfrac{\partial}{\partial y}\left(\bar{u}\overline{u'v'}\right) - \\ &\quad - \tfrac{\partial}{\partial z}\left(\bar{u}\overline{u'w'}\right) - \tfrac{\partial}{\partial x}\left[\tfrac{1}{2}\bar{u}\left(\overline{u'^2} + \overline{v'^2} + \overline{w'^2}\right)\right] + \\ &\quad + \nu\left(\bar{u}\tfrac{\partial^2\bar{u}}{\partial z^2} + \overline{u'\Delta u'} + \overline{v'\Delta v'} + \overline{w'\Delta w'}\right) \ . \end{aligned} \tag{7}$$

The first term on the right in equation (7) is related to the work done by the pressure forces per unit time; the next four terms are related to the energy flux density due to the fluid mass transfer; the last term, involving the viscosity coefficient, can be written as

$$\begin{aligned} &\nu\left(\bar{u}\tfrac{\partial^2\bar{u}}{\partial z^2} + \overline{u'\Delta u'} + \overline{v'\Delta v'} + \overline{w'\Delta w'}\right) = \\ &\quad = \nu\tfrac{\partial^2}{\partial z^2}\left(\tfrac{1}{2}\bar{u}^2\right) + \tfrac{1}{2}\nu\Delta\left(\overline{u'^2} + \overline{v'^2} + \overline{w'^2}\right) - \\ &\quad - \nu\left(\tfrac{\partial\bar{u}}{\partial z}\right)^2 - \nu\left[\overline{\left(\tfrac{\partial u'}{\partial x}\right)^2} + \overline{\left(\tfrac{\partial u'}{\partial y}\right)^2} + \overline{\left(\tfrac{\partial u'}{\partial z}\right)^2}\right] - \\ &\quad - \nu\left[\overline{\left(\tfrac{\partial v'}{\partial x}\right)^2} + \overline{\left(\tfrac{\partial v'}{\partial y}\right)^2} + \overline{\left(\tfrac{\partial v'}{\partial z}\right)^2}\right] - \\ &\quad - \nu\left[\overline{\left(\tfrac{\partial w'}{\partial x}\right)^2} + \overline{\left(\tfrac{\partial w'}{\partial y}\right)^2} + \overline{\left(\tfrac{\partial w'}{\partial z}\right)^2}\right] \ ; \end{aligned} \tag{8}$$

the first two terms on the right in equation (8) are related to the energy flux density due to internal friction (momentum transfer through collisions caused by viscosity); the remaining terms imply heat production; for an adiabatic flow they are equal to $-(1/\rho)\,\mathrm{div}\,\mathbf{q}$, where $\mathbf{q}$ is the heat flux density [2].

A similar analysis can be done for each of the equations (5) and (6) separately. The terms $-\bar{u}\frac{\partial}{\partial z}\overline{u'w'}$ on the right in equation (5) and $-\overline{u'w'}\frac{\partial\bar{u}}{\partial z}$ in equation (6) (which together give an energy flux density $-\frac{\partial}{\partial z}\left(\bar{u}\overline{u'w'}\right)$), have a special meaning when taken separately: each of them represents a coupling between the mean flow and the turbulent flow. In particular, $-\overline{u'w'}\frac{\partial\bar{u}}{\partial z}$ is a production rate of turbulence energy (while $-\bar{u}\frac{\partial}{\partial z}\overline{u'w'}$ is a production rate of mean-flow energy; "production" means here either a positive or a negative contribution).

We focus now on equation (6). The components of the Reynolds stress tensor have a small, local (finite) variation, at least for a homogeneous turbulence; assuming a homogeneous turbulence along the x, y-coordinates and taking a spatial averaging with respect to these coordinates we get from equation (6)

$$\begin{aligned} \overline{u'w'}\tfrac{d\bar{u}}{dz} \ &= \ \nu\left(\overline{u'\tfrac{d^2}{dz^2}u'} + \overline{v'\tfrac{d^2}{dz^2}v'} + \overline{w'\tfrac{d^2}{dz^2}w'}\right) - \nu C = \\ &= \tfrac{1}{2}\nu\tfrac{d^2}{dz^2}\left(\overline{u'^2} + \overline{v'^2} + \overline{w'^2}\right) - \\ &\quad - \nu\left[\overline{\left(\tfrac{du'}{dz}\right)^2} + \overline{\left(\tfrac{dv'}{dz}\right)^2} + \overline{\left(\tfrac{dw'}{dz}\right)^2}\right] - \nu C \ , \end{aligned} \tag{9}$$

where C is a positive constant arising from spatial averages of the type $u'\frac{\partial^2}{\partial x^2}u'$ (for the sake of simplicity, in equation (9) the spatial averaging is not indicated explicitly by an additional averaging sign). The first term on the right in equation (9) involves the turbulence energy, while the second term is the heat produced by internal friction (viscosity) along the z-axis. For small fluctuations these terms have a slow variation with z, at least for not too small distances from the surface; therefore, we may assume that the main contribution to the *rhs* of equation (9) is a constant. The deviations of this term from a constant are viewed as small corrections, which are neglected here; this implies a limited, intermediate range of validity for the coordinate z. Therefore, we assume a constant production rate of turbulence energy

$$\overline{u'w'}\frac{d\bar{u}}{dz} = const = -\beta u_*^2 u_1 \ , \tag{10}$$

where $u_1 = \frac{d\bar{u}}{dz}\mid_{z=0} = \bar{u}'\mid_{z=0}$. This scheme of approximation amounts to a perturbation-theoretical treatment, where the correlation functions of the fluctuating quantities (including correlation functions of derivatives of fluctuating quantities), like those involved in equation (9), are assumed to vary slowly along the z-axis. The sign of the constant in equation (10) (the sign of the product βu_1) remains undetermined. A constant production rate of turbulence energy is suggested by experimental studies, at least for not too small distances from the surface [9,10].

Equation (10) is a closure assumption, which amounts to a turbulence model; [11] using equation (4) we get

$$\frac{d\bar{u}}{dz} = -\frac{hu_1}{z-h} \tag{11}$$

and

$$\bar{u}(z) = -hu_1 \ln\left|\frac{z-h}{h}\right| \tag{12}$$

for a vanishing velocity on the surface ($\bar{u}\mid_{z=0} = 0$). This law is different from the well-known logarithmic profile $\bar{u}(z) = (u_*/k) \times \ln(z/z_0)$ [12–17], where $k \simeq 0.4$ is the von Karman constant and z_0 is a cut-off parameter [1,2]; in particular, equation (12) includes a characteristic length h and the slope of the velocity at $z = 0$. The difference arises from the fact that the governing quantities of the velocity distribution are not only the velocity u_* and the coordinate z, but the Reynolds tensor too. Various logarithmic laws have been suggested for various z-regions in turbulent boundary layers, [18–21] including similarity analysis in the presence of a pressure gradient [22]. Equation (12) gives a displaced logarithmic profile of velocity, with h the displacement height. For large values of z ($z \gg h$) we recover a logarithmic law

$$\bar{u}(z) \simeq -hu_1 \ln|z/h| \ , \ z \gg |h| \ , \tag{13}$$

while for small values of z (comparable with $|h|$) equation (12) gives a linear z-dependence $\bar{u}(z) \simeq u_1 z$; this linear dependence corresponds also to the absence of the pressure-gradient ($A = 0$ in equation (4)), where $h \to \infty$. For $h > 0$ ($\beta/A > 0$) the displaced logarithmic profile given by equation (12) corresponds to an inversion boundary layer, while for $h < 0$ ($\beta/A < 0$) the displaced logarithmic profile indicates a direct boundary layer. A positive displacement height (in an inversion layer) disrupts the logarithmic velocity profile $\ln z$ in two branches, separated by a logarithmic singularity at $z = h$, as shown by equation (12) and in Fig. 1. For $z < 2h$ the direction of the flow is opposite (negative, for $u_1 < 0$) to the direction of the main flow for $z > 2h$. In the vicinity of the surface the velocity is linear in z, $\bar{u}(z) \simeq u_1 z$ for $z \ll h$, and negative ($u_1 < 0$).

Records are known of inversion layers in Meteorology ("counter-gradient" layers) [23]. The inversion layer may also be viewed as a "sub-layer", though it is different from what is usually called a sub-layer in the logarithmic profile [2]; rather, the range of small values of z where the velocity is linear in z can be considered a

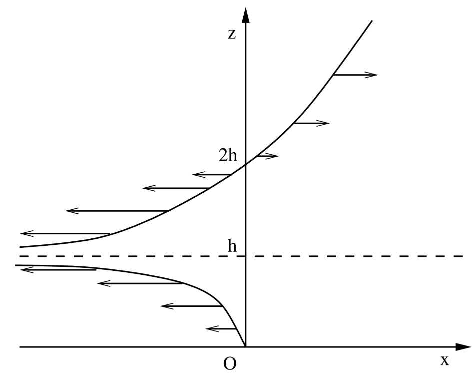

Fig. 1. Logarithmic singularity (velocity "rupture") in an inversion layer, which may occur in the turbulent boundary layer. The arrows indicate the velocity (equation (12)).

sub-layer. The "linear sub-layer" is well documented, both experimentally and theoretically [9,24]. An offset to the z-coordinate of the h-type inside the logarithm in equation (12) has been pointed out in atmospheric turbulence, especially in connection with self-similarity symmetries of the equations of motion [25–29]. Local self-similarity arguments used in getting a consistent organization of the experimental data in the turbulent structure of the atmospheric boundary layer indicated the existence of a finite parameter similar to h (the boundary-layer depth), a linear dependence on the coordinate z of the shear stress $\overline{u'w'}$, a connection between the shear stress $\overline{u'w'}$ and the gradient of the mean velocity as given in equations (6) and (9), and, especially, a constant rate of production of the turbulence energy, as expressed by equation (9), at least for not too small distances from the surface (usually for $z/|h| > 0.2$; the Coriolis force and the gravitational force are neglected in our considerations) [30–34]. A parametrization model of the planetary boundary layer in conditions of general atmospheric circulation suggests similar conclusions [35].

It is also worth noting that assuming an inhomogeneous turbulence along the x-axis with $\partial\overline{u'^2}/\partial x = const$ within the same procedure as above, equation (6) leads to

$$\overline{u'w'}\frac{d\bar{u}}{dz} = C_1\bar{u} + C_2 \ , \tag{14}$$

where $C_{1,2}$ are two constants; making use of $\overline{u'w'}$ given by equation (4), we get $\bar{u} \sim (z-h)^{C_3} + const$, where C_3 is another constant. Therefore, in this case the velocity profile is changed into a power law. Consequently, it seem that a (displaced) logarithmic profile of velocity is obtained only for homogeneous turbulence along the in-plane directions and a constant pressure-gradient. In the equilibrium equation (2) this pressure gradient is compensated by the gradient (along the vertical axis) of the Reynolds shear stress (for vanishing viscosity).

We turn now to equation (2) and include the viscosity coefficient; we are interested in the limit $\nu \to 0$. We assume a homogeneous turbulence, i.e. $A = -(1/\rho)dp/dx > 0$ ($\partial\overline{u'^2}/\partial x = 0$). For a direct boundary layer we have $\beta = -1$ and $u_1 > 0$; changing h into $-h$, we get from equations (2) and (9)

$$0 = \frac{u_*^2}{h} - \frac{d}{dz}\frac{u_*^2 u_1}{\bar{u}'} + \nu\bar{u}'' \tag{15}$$

(with the notation $A = -\frac{1}{\rho}\frac{dp}{dx} = u_*^2/h$). A first integration leads to

$$\bar{u}' = \frac{u_*^2}{2h\nu}\left[-(z+\bar{h}) \pm \sqrt{(z+\bar{h})^2 + z_h^2}\right] \ , \tag{16}$$

where

$$\bar{h} = h\left(1 - \frac{1}{R}\right) \; , \; z_h^2 = \frac{4h^2}{R} \qquad (17)$$

and $R = u_*^2/\nu u_1$ is a generalized Reynolds number; $R_c = 1$ plays the role of a critical value for the existence of a positive parameter $\bar{h}$ ($\bar{h} > 0$ for $R > R_c = 1$). Equation (11) is recovered in the limit $\nu \to 0$ ($R \to \infty$) by the sign $+$ in equation (16). The solution $\bar{u}$ can be obtained by integrating this equation with the boundary condition $\bar{u}\mid_{z=0} = 0$. We get

$$\bar{u}(z) = hu_1 \left\{ \ln \frac{z+\bar{h}+\sqrt{(z+\bar{h})^2+z_h^2}}{\bar{h}+\sqrt{\bar{h}^2+z_h^2}} - \frac{z_h^2}{2\left[z+\bar{h}+\sqrt{(z+\bar{h})^2+z_h^2}\right]^2} + \right.$$
$$\left. + \frac{z_h^2}{2\left(\bar{h}+\sqrt{\bar{h}^2+z_h^2}\right)^2} \right\} ; \qquad (18)$$

by comparing this equation with the displaced logarithmic equation (12), we can see that the only effect of the viscosity is to bring corrections of the order $1/R$ in the limit $R \gg 1$.

The modification brought by viscosity is more appreciable in the inversion boundary layer. For an inversion layer we assume $A = -(1/\rho)dp/dx > 0$, $\beta = 1$ and $u_1 < 0$; equation (2) becomes

$$0 = \frac{u_*^2}{h} + \frac{d}{dz}\frac{u_*^2 u_1}{\bar{u}'} + \nu \bar{u}'' \qquad (19)$$

with $A = -\frac{1}{\rho}\frac{dp}{dx} = u_*^2/h > 0$. A first integration leads to

$$\bar{u}' = -\frac{u_*^2}{2h\nu}\left[z - \bar{h} \pm \sqrt{(z-\bar{h})^2 + z_h^2}\right] \; , \qquad (20)$$

where $\bar{h} = h(1 - 1/R)$, $z_h^2 = 4h^2/R$ and $R = u_*^2/\nu\,|u_1|$; a positive displacement height $\bar{h}$ exists for $R > R_c = 1$. Equation (11) is recovered in the limit $\nu \to 0$ ($R \to \infty$) by

$$\bar{u}' = \begin{cases} -\frac{u_*^2}{2h\nu}\left[z - \bar{h} + \sqrt{(z-\bar{h})^2 + z_h^2}\right] \; , \; z < \bar{h} \; , \\ -\frac{u_*^2}{2h\nu}\left[z - \bar{h} - \sqrt{(z-\bar{h})^2 + z_h^2}\right] \; , \; z > \bar{h} \; ; \end{cases} \qquad (21)$$

we can see that the derivative $\bar{u}'$ has a discontinuity at $z = \bar{h}$, given by

$$\bar{u}'(\bar{h}) = sgn(z - \bar{h})\,|u_1|\sqrt{R} \; ; \qquad (22)$$

the function $\bar{u}(z)$ acquires a "sharp-edge" profile in the vicinity of $z = \bar{h}$, controlled by the Reynolds number R; for $R \to \infty$ ($\nu \to 0$) the logarithmic singularity is recovered. The solution $\bar{u}$ can be obtained by integrating equations (21) with the boundary condition $\bar{u}\mid_{z=0} = 0$. We get

$$\bar{u}(z) = -h\,|u_1| \left\{ \ln\left| \frac{\sqrt{(z-\bar{h})^2+z_h^2} - \left|z-\bar{h}\right|}{\sqrt{\bar{h}^2+z_h^2} - \bar{h}} \right| + \right.$$
$$+ \frac{1}{2z_h^2}\left[\sqrt{(z-\bar{h})^2 + z_h^2} - \left|z - \bar{h}\right|\right]^2 -$$
$$\left. - \frac{1}{2z_h^2}\left(\sqrt{\bar{h}^2 + z_h^2} - \bar{h}\right)^2 \right\} \qquad (23)$$

and

$$\bar{u}(\bar{h}) = -\frac{1}{2}h\,|u_1|\left(\ln R + 1 - \frac{1}{R}\right) \; ; \qquad (24)$$

by comparing with equation (12) (with h negative) we can see that $\sqrt{R}$ scales as $h/\,|z - h|$ for z in the vicinity of h and $R \to \infty$. For a viscous flow the inversion layer is preserved for $R > R_c = 1$ ($\nu < u_*^2/\,|u_1|$), while acquiring an edge at $\bar{h}$, which gets sharper with increasing R; an estimate for the width of the sharpness of the inversion layer is the parameter z_h, while $2\bar{h}$ can be taken as an estimate of the width of the inversion layer for large values of R. A sketch of a sharp-edged inversion layer is shown in Fig. 2. Such turnovers or cusps in wind profile have been recorded in nocturnal jets in turbulent atmospheric boundary layers [36–41]. We note also that for large values of the viscosity ($R \ll 1$) the sign of the "renormalized" height $\bar{h}$ may change in the above equations, and a transition direct-inversion layer may occur.

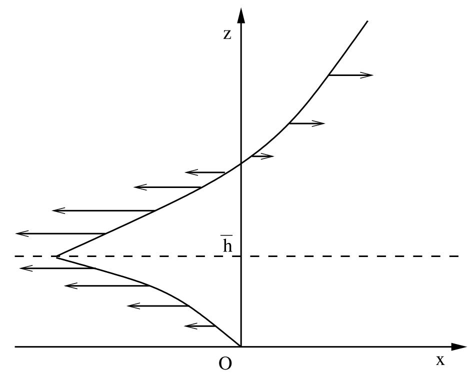

Fig. 2. A sharp-edged inversion layer (equation (23)), shown schematically.

We turn now to the temperature distribution associated with the turbulent flow in the boundary layer. We assume a homogeneous turbulence along the x, y-directions, according to the discussion above. According to our hypothesis the production rate of turbulence energy is a constant, $-\overline{u'w'}\frac{d\bar{u}}{dz} = \beta u_*^2 u_1$; for both a direct and an inversion layer the product βu_1 is negative; we can see that the turbulence energy is diminished in fact (the "production" rate of turbulence energy is in fact a "loss" rate of turbulence energy). Equation (10) gives a representation of the main contribution to the sum of the correlation functions involved in the *rhs* of equation (9), spatially averaged, in terms of our notations $-\beta u_*^2 u_1$. We note that for the opposite sign $\beta u_1 > 0$, the flow direction is reversed both in the direct and inversion layer, corresponding to an increase in the turbulence energy (this situation can also be obtained for $-(1/\rho)dp/dx < 0$). In addition, it is worth noting that there is a non-vanishing energy flux transported along the z-coordinate for large z, due to the fact that the surface is unbounded (as seen from an increasing velocity $\bar{u}(z)$ with increasing z); this is why the validity of the present results is limited to finite values of z.

According to equation (8), the equation of heat transfer is

$$T\frac{ds}{dt} = -\frac{1}{\rho}\,\mathrm{div}\,\mathbf{q} + \nu\left(\frac{\partial \bar{u}}{\partial z}\right)^2 +$$
$$+ \nu\left[\overline{\left(\frac{\partial u'}{\partial z}\right)^2} + \overline{\left(\frac{\partial v'}{\partial z}\right)^2} + \overline{\left(\frac{\partial w'}{\partial z}\right)^2}\right] + \nu C \; , \qquad (25)$$

where s is the entropy per unit mass, T is the temperature, $\mathbf{q}$ is the heat flux density and C is the same constant that appears in equation (9); the thermal conduction is given by $\mathbf{q} = -\kappa\,\mathrm{grad}\,T$, where κ is the thermal conductivity [2]. Equation (25) shows the increase in time of the entropy by internal friction and heat conduction. In general, in the *lhs* of equation (25) we take both a time averaging and a spatial averaging with respect to the in-plane coordinates; it implies both mean and fluctuating temperatures and entropies (and velocities). Since the fluctuating quantities are assumed to be small in comparison with the mean quantities, we may leave aside these contributions to equation (25). In addition, the *lhs* of equation (25) is vanishing (as for an adiabatic flow) for our particular problem (since we have only one component $\bar{u}$ of the mean velocity along the homogeneous-turbulence x-axis). Then the temperature distribution is given by

$$\kappa \frac{d^2T}{dz^2} = -\rho\nu \left(\frac{d\bar{u}}{dz}\right)^2 . \qquad (26)$$

For $\nu \to 0$ we may use equation (11) for $d\bar{u}/dz$; we get

$$T = \frac{\rho\nu u_1^2 h^2}{\kappa} \ln\left|\frac{z-h}{h}\right| + \frac{\rho\nu u_1^2 h}{\kappa} z \qquad (27)$$

with the boundary conditions $T\,|_{z=0} = \frac{dT}{dz}\,|_{z=0} = 0$. We can see that the temperature decreases with the distance from the surface; for an inversion layer the temperature has a logarithmic singularity at $z = h$; in general, the temperature distribution for an inversion layer exhibits a non-monotonous dependence on z. For the evolution of the temperature in time, we use $ds = c_p dT$ in equation (25), where c_p is the specific heat at constant pressure [2]; the solution is specific to a diffusion process. Models of temperature distributions in atmospheric boundary layers, or associated with stratified air-flow inversion layers (jets), or in convective atmospheric boundary layers have been discussed in Refs. [23,42,43].

A more general situation may appear for boundary layers in the presence of so-called "external forcings", which may generate local gradients of temperature or of fluid-mixture concentration along the z-coordinate [23]. Under such circumstances we may expect a z-dependence of the pressure p; consequently, the parameter $A = (-1/\rho)(\partial p/\partial x)$ is not a constant anymore, as assumed heretofore, but, instead, it becomes a function of z (the gravitation produces a similar effect). For instance, for a simple linear dependence of the form $A = (-1/\rho)(\partial p/\partial x) = a + 2bz$, where a and b are two (positive) constants, the procedure described in the present paper (equations (2) and (9)) leads to a quadratic z-dependence of $\overline{u'w'}$ and $1/(d\bar{u}/dz)$, of the form $\overline{u'w'} = az + bz^2 - u_*^2$, for $\beta = 1$ (inversion layer). The velocity distribution is $\bar{u} \sim \ln[|z/h_1 - 1|/(z/h_2 + 1)]$, where $h_2 > h_1 > 0$ are two parameters related to the roots of the equation $az + bz^2 - u_*^2 = 0$. We can see that the inversion layer is preserved, but the logarithmic velocity profile suffers now an appreciable change for large z, where the velocity tends to a constant $\sim \ln(h_2/h_1)$ (we remind that the derivations made here are not valid for large z). A similar result is obtained for $b < 0$ (gravitation effect), where the velocity profile $\bar{u} \sim \ln|(z/h_1 - 1)/(z/h_2 - 1)|$, $h_2 > h_1 > 0$, exhibits a "scissors-like", double inversion layer (stratified layers).

The logarithmic singularity of the inversion layer at $z = h$ or the discontinuity in the slope of the function $\bar{u}(z)$ at $z = \bar{h}$ might be viewed as unsatisfactory features of the present model. However, $\overline{u'w'} \sim z - h$ given by equation (4) can be considered as a first-order contribution in powers of z; an additional, next-order z^2-term could lead to $\overline{u'w'}$ of the form $\overline{u'w'} \sim (z-h)^2 + \varepsilon^2$, where ε is a small parameter. Indeed, it seems that the condition of vanishing $\overline{u'w'}$ at $z = h$ is too strong. Similarly, the constant production rate of turbulence energy $\overline{u'w'}\frac{d\bar{u}}{dz} = -u_*^2 u_1$ (equation (9)) may acquire an additional z-term, leading to $\overline{u'w'}\frac{d\bar{u}}{dz}$ of the form $\overline{u'w'}\frac{d\bar{u}}{dz} \sim z - h'$ (where h' is, in general, different from h). Then, $\bar{u}'$ behaves essentially as $\bar{u}' \sim (z-h)/[(z-h)^2 + \varepsilon^2]$ and $\bar{u} \sim \ln[(z-h)^2 + \varepsilon^2]$, which might be a more acceptable functional dependence on z of the velocity. However, derivation of higher-order terms in the shear stress and the production rate of turbulence energy requires further, additional model assumptions.

In conclusion, we may say that a displaced logarithmic profile of velocity has been derived here for the boundary layer in the turbulent flow over an unbounded flat surface by assuming a constant pressure-gradient and a constant production rate of turbulence energy, associated with a homogeneous turbulence along the in-plane directions. For a direct boundary layer the displacement height is negative, while for an inversion boundary layer the displacement height is positive. The inversion layer splits the logarithmic profile in two branches, separated by a logarithmic singularity (a "rupture" in the velocity distribution). In the inversion layer the fluid flows in direction opposite to the main flow. In a viscous flow the singularity in the inversion layer becomes a sharp edge controlled by a generalized Reynolds number R (small values of the viscosity bring only corrections of the order $1/R \ll 1$ for the direct boundary layer). The associated temperature distribution has also been estimated for such boundary layers. It has been shown that gravitation and atmospheric, local, thermal or fluid-mixture concentration gradients ("external forcings") can bring various modifications to the inversion layers. The procedure employed in this paper of treating the turbulent boundary layer is based on the Reynolds equations and on certain assumptions regarding the components of the Reynolds stress tensor, derived from the assumption of homogeneous turbulence, or the symmetry of the problem. The Reynolds equations remain, in general, undetermined, as well as some features of the boundary conditions; for instance, the distinction criterion between a direct boundary layer and an inversion one (the choice of the parameter $\beta = \pm 1$), or the choice of the sign of the product βu_1 (as well as the parameter u_*^2) remain undetermined; there appears to be no obvious, physical, quantitative criteria to discriminate between the two types of boundary layers, the difference being made by the boundary conditions, and, probably, the (undetermined) preparation conditions of the turbulence.

Finally, we include here a brief description of a similar situation for a pipe with a circular cross-section of radius R. For a uniform flow with mean velocity $u(r)$ along the z-axis of the pipe the Reynolds shear stress is

$$2r\overline{v_r' v_z'} = \frac{\beta u_*^2}{h}\left(r^2 - R^2 - 2hR\right) ,$$

where $h = \beta u_*^2/A$, $A = -(1/\rho)(dp/dz) = const > 0$ (r being the radial coordinate). The constant production rate of turbulence energy is $-\overline{v_r' v_z'}(du/dr) = -\beta u_*^2 u_1$, where $u_1 = -\frac{du}{dr}\,|_{r=R}$ and the velocity distribution is given by

$$u(r) = hu_1 \ln\left|\frac{r^2 - R^2 - 2hR}{2hR}\right|$$

for $u(R) = 0$ (and zero viscosity). For $\beta = +1$ and $u_1 > 0$ the boundary layer is a direct one, and the viscosity brings only small corrections in the limit $\nu \to 0$. For $\beta = -1$ and $u_1 < 0$ the parameter h is negative and we have an inversion layer; the velocity exhibits a logarithmic singularity at $r = (R^2 - 2|h|R)^{1/2}$ (the flow is possible for $R > 2|h|$), which is made finite by viscosity.

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